

Note

On the Number of Unique Subgraphs of a Graph*

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A question of Entringer and Erdős concerning the number of unique subgraphs of a graph is answered.

Entringer and Erdős [1] call a subgraph H of a graph G unique if H is not isomorphic to any other subgraph of G . If $f(n)$ is the largest number of unique subgraphs a graph on n vertices can have, they prove

$$\log f(n) > \frac{1}{2}n^2 - cn^{3/2}$$

for $c > 3/\sqrt{2}$ and n sufficiently large. (Here and in the sequel all logarithms have base 2.) Harary and Schwenk [3] improved this bound and obtained $\log f(n) > \frac{1}{2}n^2 - cn \cdot \log n$ for $c > 2$ and n sufficiently large.

It will be proved below that

$$\log f(n) = \frac{1}{2}n^2 - n \cdot \log n + O(n).$$

Since the number of nonisomorphic graphs on n vertices is

$$(2^{\binom{n}{2}}/n!)[1 + (n^2 - n)/(2^n - 1) + O(n^3/2^{3n/2})]$$

(see e.g. [2, p. 196]), we have

$$\log f(n) \leq \frac{1}{2}n^2 - n \cdot \log n + O(n).$$

On the other hand, given n we construct a graph G_n on n vertices with $2^{\frac{1}{2}n^2 - n \cdot \log n + O(n)}$ unique subgraphs as follows:

Let $m = \lceil \log n \rceil$ and $N = n - m - 2$. Then $N \leq 2^m - m - 1$.

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Let $G_n = A \cup B \cup C$ where

$A = K_N$, the complete graph on N points,

B is a rigid tree with m vertices (such a tree exists for each $m \geq 7$ (e.g. see [1])),

$C = K_2$, a single edge connecting points c_0 and c_1 ,
connected as follows:

G_n contains all edges (c_1, b) for $b \in B$ and no other edges between C and $A \cup B$; furthermore, if we view A as a set of subsets of B each containing at least two points (which is possible since $N \leq 2^m - m - 1$), then G_n contains the edge (a, b) where $a \in A$ and $b \in B$ if and only if $b \in a$.

Now define the subgraph H_n of G_n as follows:

$H_n = A' \cup B \cup C$ where A' is the vertex graph on N vertices (that is, A' is totally disconnected) and A' , B , C are interconnected like A , B , C in G_n . That is, H_n contains the same n points as G_n but has $\binom{N}{2}$ edges less.

If H is a subgraph of G_n such that $H_n \subset H \subset G_n$ then H is unique: First, c_0 is the only point of H with degree one, and therefore if we imbed H in G_n the point c_0 of H must go to the point c_0 of G_n . Next it follows that c_1 must go to c_1 and therefore that $B \subset H$ must map onto $B \subset G$. Since B is rigid the imbedding restricted to B must be the identity on B . Finally, since each point of A is coded by a subset of B , A too cannot be imbedded in any other way. Therefore H is unique.

Since the number of subgraphs H between H_n and G_n is $2^{\binom{N}{2}} = 2^{\frac{1}{2}n^2 - n \cdot \log n + O(n)}$, this proves our assertion.

REFERENCES

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